

# A Euclidean Ramsey result in the plane

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## Abstract

An old question in Euclidean Ramsey theory asks, if the points in the plane are red-blue coloured, does there always exist a red pair of points at unit distance or five blue points in line separated by unit distances? An elementary proof answers this question in the affirmative.

## 1 Introduction

Many problems in Euclidean Ramsey theory ask, for some  $d \in \mathbb{Z}^+$ , if  $E^d$  is coloured with  $r \geq 2$  colours, does there exist a colour class containing some desired geometric structure? Research in Euclidean Ramsey theory was surveyed in [2–4] by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus; for a more recent survey, see Graham [5].

Say that two geometric configurations are congruent iff there exists an isometry (distance preserving bijection) between them. For  $d \in \mathbb{Z}^+$ , and geometric configurations  $F_1, F_2$ , let the notation  $\mathbb{E}^d \rightarrow (F_1, F_2)$  mean that for any red-blue coloring of  $\mathbb{E}^d$ , either the red points contain a congruent copy of  $F_1$ , or the blue points contain a congruent copy of  $F_2$ . For a positive integer  $i$ , denote by  $\ell_i$  the configuration of  $i$  collinear points with distance 1 between consecutive points. One of the results in [3] states that

$$\mathbb{E}^2 \rightarrow (\ell_2, \ell_4). \quad (1)$$

In the same paper, it was asked if  $\mathbb{E}^2 \rightarrow (\ell_2, \ell_5)$ , or perhaps a weaker result holds:  $\mathbb{E}^3 \rightarrow (\ell_2, \ell_5)$ .

The result (1) was generalised by Juhász[7], who proved that if  $T_4$  is any configuration of 4 points, then  $\mathbb{E}^2 \rightarrow (\ell_2, T_4)$ . Juhász (personal communication, 10 February 2017) informed the author that Iván’s thesis [6] contains a proof that for any configuration  $T_5$  of 5 points,  $\mathbb{E}^3 \rightarrow (\ell_2, T_5)$  (which implies that  $\mathbb{E}^3 \rightarrow (\ell_2, \ell_5)$ ). Arman and Tsaturian[1] proved that  $\mathbb{E}^3 \rightarrow (\ell_2, \ell_6)$ .

In this paper, it is proved that  $\mathbb{E}^2 \rightarrow (\ell_2, \ell_5)$ :

**Theorem 1.1.** *Let the Euclidean space  $\mathbb{E}^2$  be coloured in red and blue so that there are no two red points distance 1 apart. Then there exist five blue points that form an  $\ell_5$ .*

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## 2 Proof of Theorem 1.1

The proof is by contradiction; it is assumed that there are no five blue points forming an  $\ell_5$ . The following lemmas are needed.

**Lemma 2.1.** *Let  $\mathbb{E}^2$  be coloured in red and blue so that there is no red  $\ell_2$ . If there is no blue  $\ell_5$ , then there are no three blue points forming an equilateral triangle with side length 3 and with a red centre.*

*Proof.* Suppose that  $\mathbb{E}^2$  is coloured in red and blue so that there is no red  $\ell_2$  and no blue  $\ell_5$ . Suppose that blue points  $A$ ,  $B$  and  $C$  form an equilateral triangle with side length 3 and with red centre  $O$ . Consider the part of the unit triangular lattice shown in Figure 1(a). The points  $D$ ,  $E$ ,  $F$ ,  $G$  are blue, since they are distance 1 apart from  $O$ . The point  $X$  is red; otherwise  $XADEB$  is a red  $\ell_5$ . Similarly,  $Y$  is red (to prevent red  $YAFGC$ ). Then  $X$  and  $Y$  are two red points distance 1 apart, which contradicts the assumption.  $\square$

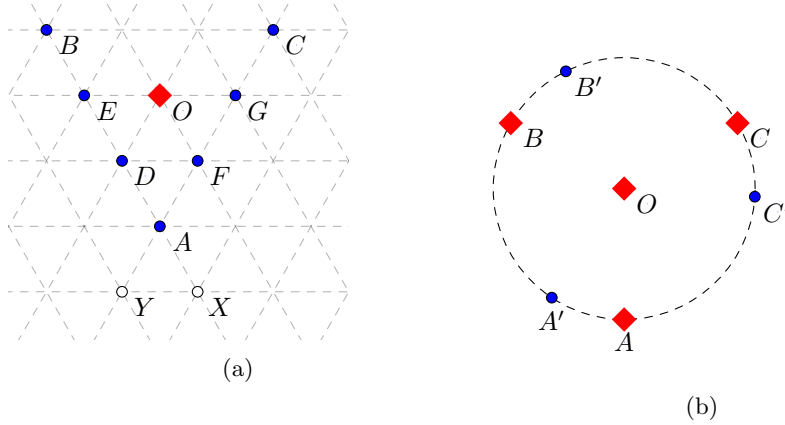


Figure 1: Red points are denoted by diamonds, blue points are denoted by discs.

**Lemma 2.2.** *Let  $\mathbb{E}^2$  be coloured in red and blue so that there is no red  $\ell_2$ . If there is no blue  $\ell_5$ , then there are no three red points forming an equilateral triangle with side length  $\sqrt{3}$  and with a red centre.*

*Proof.* Suppose that  $\mathbb{E}^2$  is coloured in red and blue so that there is no red  $\ell_2$  and no blue  $\ell_5$ . Suppose that blue points  $A$ ,  $B$  and  $C$  form an equilateral triangle with side length 3 and with red centre  $O$ . Let  $A'$ ,  $B'$ ,  $C'$  be the images of  $A$ ,  $B$  and  $C$ , respectively, under a rotation about  $O$  so that  $AA' = BB' = CC' = 1$  (see Figure 1(b)). Then  $A'$ ,  $B'$ ,  $C'$  are blue and form an equilateral triangle with side length  $\sqrt{3}$  and red center  $O$ , which contradicts the result of Lemma 2.1.  $\square$

Define  $\mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5, \mathfrak{T}_6, \mathfrak{T}_7$  to be the configurations of three, four, five, six and seven points (respectively), depicted in Figure 2 (all the smallest distances between the points are equal to  $\sqrt{3}$ ).

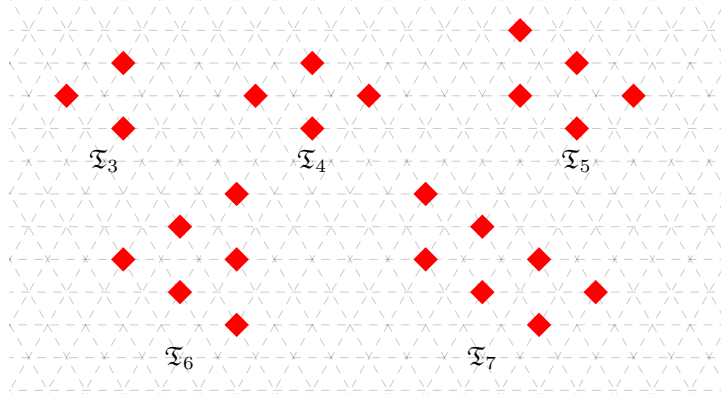


Figure 2

**Lemma 2.3.** *Let  $\mathbb{E}^2$  be coloured in red and blue so that there is no red  $\ell_2$ . If there is no blue  $\ell_5$ , then there are no seven red points forming a  $\mathfrak{T}_7$ .*

*Proof.* Suppose that  $\mathbb{E}^2$  is coloured in red and blue so that there is no red  $\ell_2$  and no blue  $\ell_5$ . Suppose that  $A, B, C, D, E, F$  and  $G$  are red points forming a  $\mathfrak{T}_7$  (as in Figure 3). Let  $X$  be the reflection of  $F$  in  $BC$ . Let  $X', A', F'$  be the images of  $X, A, F$ , respectively, under the clockwise rotation about  $B$  such that  $XX' = AA' = FF' = 1$ . Since  $A$  and  $F$  are red,  $A'$  and  $F'$  are blue. If  $X'$  is blue, then  $X'A'F'$  is a blue equilateral triangle with side length 3 and red center  $B$ , which contradicts the result of Lemma 2.1. Therefore,  $X'$  is red. Let  $X'', D'', F''$  be the images of  $X, D, F$ , respectively, under the clockwise rotation about  $C$  such that  $XX'' = DD'' = FF'' = 1$ . Since  $D$  and  $F$  are red,  $D''$  and  $F''$  are blue. If  $X''$  is blue, then  $X''D''F''$  is a blue equilateral triangle with side length 3 and red center  $C$ , which contradicts the result of Lemma 2.1. Therefore,  $X''$  is red. Since  $X'$  can be obtained from  $X''$  by the clockwise rotation through  $60^\circ$  about  $X$ ,  $XX'X''$  is a unit equilateral triangle, hence  $X'X''$  is a red  $\ell_2$ , which contradicts the assumption of the lemma.  $\square$

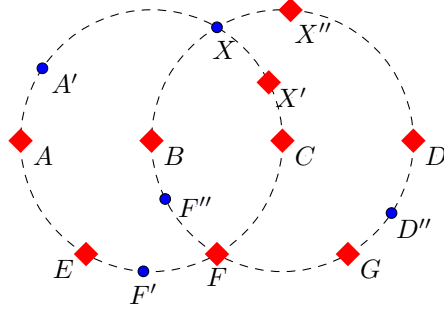


Figure 3

**Lemma 2.4.** *Let  $\mathbb{E}^2$  be coloured in red and blue so that there is no red  $\ell_2$ . Let  $A, B, C$  be three red points forming a  $\mathfrak{T}_3$ . If there is no blue  $\ell_5$ , then there exists a red  $\mathfrak{T}_6$  that contains  $\{A, B, C\}$  as a subset.*

*Proof.* Suppose that  $\mathbb{E}^2$  is coloured in red and blue so that there is no red  $\ell_2$  and no blue  $\ell_5$ . Let  $A, B, C$  be three red points forming a  $\mathfrak{T}_3$ . Consider the unit triangular lattice depicted in Figure 4.

Suppose that there is no red point  $D$  such that  $A, B, C, D$  form a  $\mathfrak{T}_4$ . Then points  $X, Y, Z$  are blue. Points  $E, F, G, H, I, J$  are blue, since each of them is distance 1 apart from a red point. If the point  $K$  is red, then the points  $L$  and  $M$  are blue and  $LMYGH$  is a blue  $\ell_5$ . Therefore,  $K$  is blue. Then  $N$  is red (otherwise  $KJIZN$  is a blue  $\ell_5$ ), hence  $P$  and  $Q$  are blue, which leads to a blue  $\ell_5$   $PQFEX$ . A contradiction is obtained, therefore there exists a red point  $D$  such that  $A, B, C, D$  form a  $\mathfrak{T}_4$ .

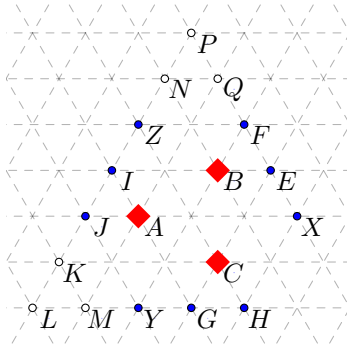


Figure 4

Let  $A, B, C, D$  form a red  $\mathfrak{T}_4$ . Consider the part of the unit triangular lattice depicted in Figure 5. Suppose that there is no red point  $E$  such that  $A, B, C, D, E$  form a  $\mathfrak{T}_5$ . Then the points  $X, F$  and  $G$  are blue. Points  $H, I, K, L, M, N$  are blue, since each of them is distance 1 apart from a red point. Point  $P$  is red (otherwise  $FHIGP$  is a blue  $\ell_5$ ), therefore  $Q$  and  $R$  are blue. Then  $X, N, M, Q, R$  form a blue  $\ell_5$ , which gives a contradiction. Hence, there exists a red point  $E$  such that  $A, B, C, D, E$  form a  $\mathfrak{T}_5$ .

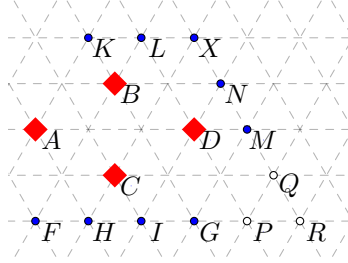


Figure 5

Let  $A, B, C, D, E$  form a  $\mathfrak{T}_5$  (Figure 6). Suppose that  $F$  is blue. By Lemma 2.2, points  $X$  and  $Y$  are blue (otherwise  $X, E, C$  ( $Y, A, D$ ) form a red triangle with side length 3 and red center  $B$ ). Points  $G, H, I, J, K, L, M, N$  are blue, since each one of them is at distance 1 from a red point. If point  $P$  is blue, then  $Q$  is red (otherwise  $QPKLF$  is a blue  $\ell_5$ ),  $U$  and  $T$  are blue and form a blue  $\ell_5$  with points  $G, H$  and  $X$ . Therefore,  $P$  is red. Similarly,  $R$  is red (otherwise  $S$  is red and  $VWJIY$  is a blue  $\ell_5$ ). Then  $A, B, C, D, E, P$  and  $R$  form a red  $\mathfrak{T}_7$ , which is not possible by Lemma 2.3. Therefore,  $F$  is red and  $A, B, C, D, E, F$  form a red  $\mathfrak{T}_6$ .

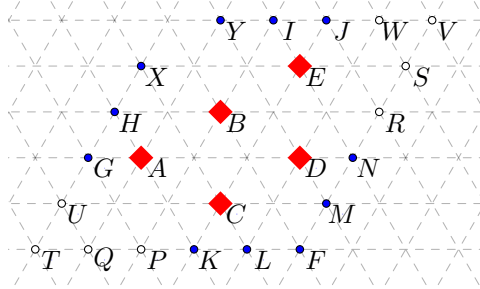


Figure 6

□

**Lemma 2.5.** *Let  $\mathbb{E}^2$  be coloured in red and blue so that there is no red  $\ell_2$ . Let  $\mathfrak{L}$  be a unit triangular lattice that contains three red points forming a  $\mathfrak{T}_3$ . If there is no blue  $\ell_5$ , then the colouring of  $\mathfrak{L}$  is unique (up to translation or rotation by a multiple of  $60^\circ$ ), and is depicted in Figure 7.*

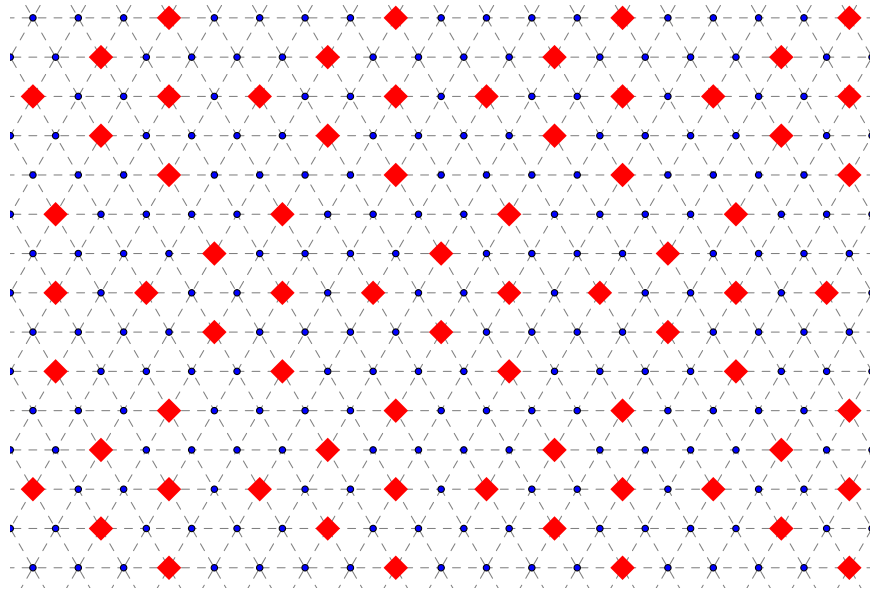


Figure 7

*Proof.* Suppose that  $\mathbb{E}^2$  is coloured in red and blue so that there is no red  $\ell_2$  and no blue  $\ell_5$ . Suppose there exist three red points of  $\mathfrak{L}$  that form a  $\mathfrak{T}_3$ . By Lemma 2.4, it may be assumed that there is a red  $\mathfrak{T}_6$ . Denote its points by  $A, B, C, D, E, F$  (see Figure 8). It will be proved that the translate  $A'B'C'D'E'F'$  of  $ABCDEF$  by the vector of length 5 collinear to  $\overrightarrow{AD}$  is red.

Consider the points shown in Figure 8. Since  $A, D$  and  $F$  are red, by Lemma 2.2,  $I$  is blue. Since  $C, F$  and  $D$  are red, by Lemma 2.2,  $J$  is blue. Points  $K, L, M, N$  are blue, since each one is distance 1 apart from a red point. If  $R$  is red, then both  $P$  and  $Q$  are blue and form a blue  $\ell_5$  with  $K, L$  and  $I$ . Therefore  $R$  is blue. Then the point  $A'$  is red (otherwise  $A'JNMR$  is a red  $\ell_5$ ).

Since  $S_1, S_2, S_3, S_4$  are blue (as distance 1 apart from red points  $D$  and  $A'$ ),  $B'$  is red. Similarly,  $F'$  is red. Points  $V$  and  $W$  are blue as they are distance 1 apart from  $C$ . Points  $U$  is blue by Lemma 2.2 (since  $A, D$  and  $B$  are red). If  $X$  is red, then  $X_1$  and  $X_2$  are blue and a blue  $\ell_5$   $UVWX_1X_2$  is formed. Therefore,  $X$  is blue. Similarly,  $Y$  is blue. By Lemma 2.4,  $A'B'F'$  must be contained in a red  $\mathfrak{T}_6$ , and since  $X$  and  $Y$  are blue, the only possible such  $\mathfrak{T}_6$  is  $A'B'C'D'E'F'$ . Hence,  $A', B', C', D', E', F'$  are blue.

Similarly, the translates of  $ABCDEF$  by vectors of length 5 collinear to  $\overrightarrow{EB}$

and  $\overrightarrow{CF}$  are red. By repeatedly applying the same argument to the new red translates, it can be seen that all the translates of  $ABCDEF$  by a multiple of 5 in  $\mathcal{L}$  are red. All the other points are blue, as each one is distance 1 apart from a red point. Hence, the colouring as in Figure 7 is obtained.  $\square$

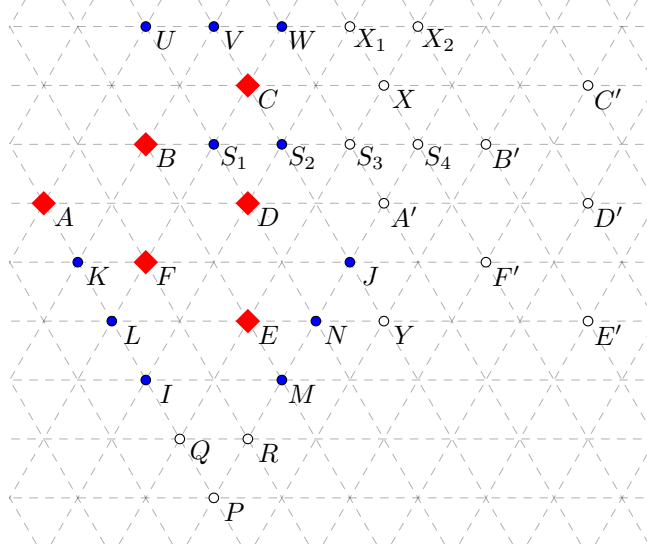


Figure 8

**Lemma 2.6.** *Let  $\mathbb{E}^2$  be coloured in red and blue so that there is no red  $\ell_2$ . Let  $\mathcal{L}$  be a unit triangular lattice that does not contain three red points forming a  $\mathfrak{T}_3$ . If there is no blue  $\ell_5$ , then the colouring of  $\mathcal{L}$  is unique (up to translation or rotation by a multiple of  $60^\circ$ ), and is depicted in Figure 9.*

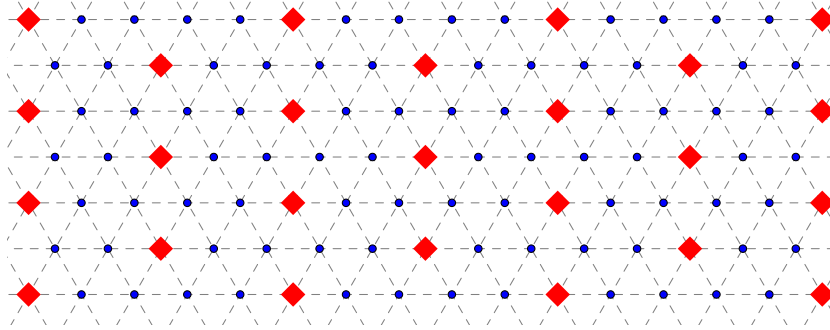


Figure 9

*Proof.* Suppose that  $\mathbb{E}^2$  is coloured in red and blue so that there is no red  $\ell_2$  and no blue  $\ell_5$ . If  $\mathfrak{L}$  does not contain a red point, then any  $\ell_5$  is blue, therefore  $\mathfrak{L}$  contains a red point  $A$ . By Lemma 2.1, one of the points of  $\mathfrak{L}$  at distance  $\sqrt{3}$  to  $A$  is red (otherwise the three such points form a blue triangle with side length 3 and red centre  $A$ ). Denote this point by  $B$  (Figure 10). Since  $\mathfrak{L}$  does not contain a red  $\mathfrak{T}_3$ , the points  $D$  and  $G$  are blue. Points  $E, F, I, H, K, J$  are blue, since they are distance 1 apart from  $B$ . Then the point  $B'$  is red (otherwise blue  $\ell_5$   $DEFG B'$  is formed). Point  $N$  is 1 apart from  $B'$ , hence blue. Then  $C$  and  $A'$  are red (otherwise a blue  $\ell_5$  is formed).

By repeating the same argument for points  $B$  and  $C$ ,  $B$  and  $A$  (instead of  $A$  and  $B$ ), and so on, it can be shown that any node of  $\mathfrak{L}$  on the line  $AB$  is red. Similarly, since  $A'$  and  $B'$  are both red, any node of  $\mathfrak{L}$  on the line  $A'B'$  is red. By the same argument,  $A'', B''$  and any node on the line containing them is red;  $A''', B'''$  and any node on the line containing them is red, and so on. By colouring all point distance 1 apart from red points blue, the colouring in Figure 9 is obtained.  $\square$

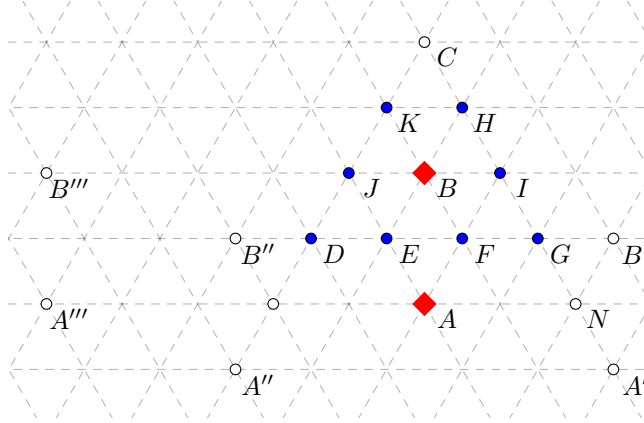


Figure 10

*Proof of Theorem 1.1.* Let the Euclidean space  $\mathbb{E}^2$  be coloured in red and blue so that there are no two red points distance 1 apart. Suppose that there are no five blue points that form an  $\ell_5$ . Then there is a red point  $A$ . Consider two points  $B$  and  $C$ , both distance 5 apart from  $A$ , such that  $|BC| = 1$ . At least one of the points  $B$  and  $C$  (say,  $B$ ) is blue. Consider the unit triangular lattice  $\mathfrak{L}$  that contains  $A$  and  $B$ . By Lemma 2.5 and Lemma 2.6,  $\mathfrak{L}$  is coloured either as in Figure 7 or as in Figure 9. But neither one of the colourings contains two points of different colour distance 5 apart, which gives a contradiction. Therefore, there exist five blue points that form an  $\ell_5$ .  $\square$



### 3 Acknowledgements

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